

EXAMPLE

$$W = z^* \quad (\text{complex conjugate})$$

$$U = x \quad V = -y$$

$$\frac{\partial U}{\partial x} = +1 \quad \frac{\partial V}{\partial y} = -1$$

3. *Integration.* The integral

$$\int_{z_1}^{z_2} f(z) dz$$

is a line integral which depends in general on the path followed from z_1 to z_2 (Figure A-7). However, the integral will be the same for two paths if $f(z)$ is regular in the region bounded by the paths. An equivalent statement is *Cauchy's theorem*:

$$\oint_C f(z) dz = 0 \quad (\text{A-9})$$

if C is any closed path lying within a region in which $f(z)$ is regular. A kind of converse is also true; if $\oint_C f(z) dz = 0$ for every closed path C within a region R , where $f(z)$ is continuous and single valued, then $f(z)$ is regular in R .

4. If $f(z)$ is regular in a region, its derivatives of all orders exist and are regular there.

5. If $f(z)$ is regular in a region R , the value of $f(z)$ at any point within R may be expressed by *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{\zeta - z} \quad (\text{A-10})$$

where C is any closed path within R encircling z once in the counterclockwise direction. This formula follows directly from the theorem of residues,

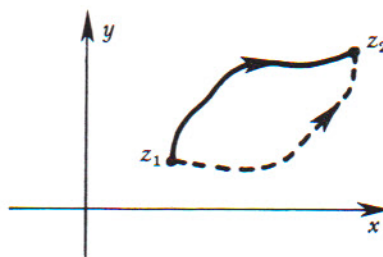


Figure A-7 Paths of integration in the complex plane

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An equivalent statement

(A-9)

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single valued, then $f(z)$ is

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(A-10)

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complex plane

item 8 below. The remarkable property of analytic functions implied by Eq. (A-10) should be noted. The values of an analytic function throughout a region are completely determined by the values of the function on the boundary of that region. See Section 5-2 for an application of this property.

Cauchy's formula may be differentiated any number of times to obtain

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \quad (\text{A-11})$$

6. A power series expansion (Taylor's series) is possible about any point z_0 within a region where $f(z)$ is regular:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$a_0 = f(z_0) \quad a_n = \frac{1}{n!} f^{(n)}(z_0) \quad (\text{A-12})$$

The region of the z -plane in which the series converges is a circle. This *circle of convergence* extends to the nearest singularity of $f(z)$, that is, to the nearest point where $f(z)$ is not analytic.

The converse is also true. Any power series convergent within a circle R represents a regular function there.

7. *The Laurent expansion.* If $f(z)$ is regular in an annular region between two concentric circles with center z_0 , then $f(z)$ may be represented within this region by a *Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where the coefficients a_n are

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{n+1}} \quad (\text{A-13})$$

C is any closed path encircling z_0 counterclockwise within the annular region. Note that the coefficient a_{-1} is

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz \quad (\text{A-14})$$

If $f(z)$ is regular in the annulus, no matter how small we make the inner circle, and yet $f(z)$ is not regular throughout the larger circle, we say that z_0 is an *isolated singularity* of $f(z)$. For such an isolated singularity, there are three possibilities:

- (a) The Laurent series for $f(z)$ may contain *no* terms with negative powers of $(z - z_0)$. This is a trivial case, and is called a *removable singularity*. By redefining $f(z)$ at the point $z = z_0$, the singularity may be removed. For example, the function

$$f(z) = \begin{cases} z & |z| > 0 \\ 1 & z = 0 \end{cases}$$

has a removable singularity at $z = 0$.

- (b) The Laurent series for $f(z)$ may contain a *finite number* of terms with negative powers of $(z - z_0)$. In this case z_0 is called a *pole of order m* , where $-m$ is the lowest power of $(z - z_0)$ appearing in the Laurent series. For example, the function $f(z) = (1/\sin z)^2$ has poles of order two at $z = 0, \pm\pi, \pm2\pi, \dots$. If $f(z)$ has a pole of order m at z_0 , the function $(z - z_0)^m f(z)$ is regular in the neighborhood of z_0 .
- (c) The Laurent series for $f(z)$ may contain *infinitely many* terms with negative powers of $(z - z_0)$. In this case, $f(z)$ is said to have an *essential singularity* at $z = z_0$. For example, $e^{1/z}$ has an essential singularity at $z = 0$ (and therefore e^z has an essential singularity at $z = \infty$).

If z_0 is an isolated singularity, the coefficient a_{-1} in the Laurent expansion is called the *residue* of $f(z)$ at z_0 . It has special importance, because of the relation (A-14), as will now be discussed.

8. The *theorem of residues* allows us to evaluate easily the integral of a function $f(z)$ along a closed path C such that $f(z)$ is regular in the region bounded by C except for a finite number of poles and (isolated) essential singularities in the interior of C . By Cauchy's theorem, the path, or *contour*, C may be deformed without crossing any singularities until it is reduced to little circles surrounding each singular point. The integral around each little circle is then given by (A-14), so that we have the theorem of residues

$$\int_C f(z) dz = 2\pi i \sum \text{residues} \quad (\text{A-15})$$

where the sum is over all the poles and essential singularities inside C . This theorem is of enormous practical importance in the evaluation of integrals, and a number of examples of its application are given in Section 3-3.

What if a pole lies on the contour? The first thing to do is to look into the physics of the problem to see if this awkward location of the pole results from some approximation. If so, one can decide on which side of the path the pole really lies and thus see whether its residue should be included or not.

A mathematical integral with a pole on the contour strictly does not exist, but, for a simple pole on the real axis, one defines the *Cauchy principal value* as

$$P \int_a^b \frac{f(x)}{x - x_0} dx = \lim_{\delta \rightarrow 0} \left[\int_a^{x_0 - \delta} \frac{f(x)}{x - x_0} dx + \int_{x_0 + \delta}^b \frac{f(x)}{x - x_0} dx \right] \quad (\text{A-16})$$

where δ is positive.

terms with negative powers
ed a *removable singularity*.
ngularity may be removed.

finite number of terms with
is called a *pole of order m*,
appearing in the Laurent
 $1/\sin z)^2$ has poles of order
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$$\int_{x_0+\delta}^b \frac{f(x)}{x-x_0} dx \quad (A-16)$$

The path for the Cauchy principal value integral can form part of a closed contour in which the ends $x_0 \pm \delta$ are joined by a small semicircle centered at the pole (see Figure A-8). Along this semicircle the integral is easy to evaluate; if we let the radius approach zero, $f(z) \rightarrow a_{-1}(z-x_0)^{-1}$. Let

$$z - x_0 = re^{i\theta} \quad dz = ire^{i\theta} d\theta$$

Then

$$\int_{\text{semicircle}} f(z) dz \rightarrow - \int_0^\pi a_{-1} i d\theta = -\pi i a_{-1}$$

and if, as is usually the case, the large semicircle gives no contribution,

$$\oint_C f(z) dz = P \int f(z) dz - \pi i (\text{residue at } z_0) \\ = 2\pi i (\sum \text{residues inside } C)$$

This gives the result

$$P \int f(z) dz = 2\pi i (\frac{1}{2} \text{ residue at } x_0 + \sum \text{residues inside } C) \quad (A-17)$$

Thus the Cauchy principal value is the average of the two results obtained with the pole inside and outside of the contour.

We often have an integral along the real axis with a simple pole just above (or just below) the axis at x_0 . We may consider the pole to be on the axis if we make the path of integration miss the pole by going around x_0 on a little semicircle below (or above). Then it follows by reasoning similar to that leading to (A-17) that the integral may be expressed in terms of the Cauchy principal value as follows:

$$\int \frac{f(x)}{x - x_0 \mp i\epsilon} dx = P \int \frac{f(x)}{x - x_0} dx \pm i\pi f(x_0)$$

We may express this result in the somewhat symbolic form

$$\frac{1}{x - x_0 \mp i\epsilon} = P \frac{1}{x - x_0} \pm i\pi \delta(x - x_0) \quad (A-18)$$

where $\delta(x - x_0)$ is the Dirac delta-function defined in (4-19).

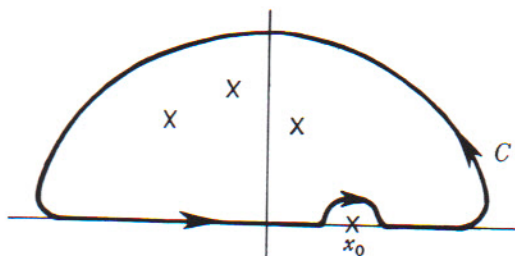


Figure A-8 Illustration of a pole on the real axis

9. The *identity theorem* states that if two functions are each regular in a region R , and have the same values for all points within some subregion or for all points along an arc of some curve within R , or even for a denumerably infinite number of points having a limit point within R , then the two functions are identical everywhere in the region. For example, if $f(z) = 0$ all along some arc in R , then $f(z)$ is the regular function 0 everywhere in R .

This theorem is useful in extending into the complex plane functions defined on the real axis. For example,

$$e^z = 1 + z + \frac{1}{2!} z^2 + \dots$$

is the unique function $f(z)$ which is equal to e^x on the real axis.

10. Consider a function $f(z)$ which is analytic in a region R of the complex plane, and assume that a finite part of the real axis is included in R . If the function $f(z)$ assumes only *real* values on that part of the real axis in R , then it can be shown that $f(z^*) = [f(z)]^*$ throughout R . That is, going from a point z to its "image" in the real axis, namely, z^* , just carries the *value* f of the function over into *its* image f^* . This is known as the *Schwartz reflection principle*.

The identity theorem forms the basis for the procedure of *analytic continuation*. A power series about z_1 represents a regular function $f_1(z)$ within its circle of convergence, which extends to the nearest singularity. If an expansion of this function is made about a new point z_2 , the resulting series will converge in a circle which may extend beyond the circle of convergence of $f_1(z)$. The values of $f_2(z)$ in the extended region are uniquely determined by $f_1(z)$ —in fact, by the values of $f_1(z)$ in the common region of convergence of $f_1(z)$ and $f_2(z)$. $f_2(z)$ is said to be the analytic continuation of $f_1(z)$ into the new region. This process may be repeated (with limitations mentioned below) until the entire plane is covered except for singular points by these *elements* of a single function $F(z)$.

EXAMPLE

$$f_1(z) = 1 + z + z^2 + z^3 + \dots$$

converges in a circle of radius 1 to

$$F(z) = \frac{1}{1-z}$$

But $F(z)$ is analytic everywhere except at the simple pole $z = 1$, and no other

ons are each regular within some subregion or even for a denumerable set within R , then the two functions are identical in R . For example, if $f(z) = 0$ in a region R , then $f(z) = 0$ everywhere in R .

plane functions defined

real axis. region R of the complex plane is included in R . If the real axis is in R , then the function $f(z)$ is analytic in R . That is, going from a point z_0 in R to a point z_1 in R carries the value f of the function $f(z)$ from $f(z_0)$ to $f(z_1)$. The Schwarz reflection principle states that if $f(z)$ is analytic in a region R of the complex plane and if $f(z)$ is real on the real axis, then $f(\bar{z}) = \overline{f(z)}$ in R .

e of analytic continuation of $f_1(z)$ within its circle of convergence. If an expansion of $f_1(z)$ in a power series will converge in a region R_2 that overlaps the circle of convergence of $f_1(z)$, then the function $f_1(z)$ is continued into R_2 by the series $f_2(z)$ —in the region R_2 the function $f_2(z)$ is analytic and $f_2(z) = f_1(z)$ in the common region. The process is continued until the entire plane is covered by the elements of a single

$z = 1$, and no other

function analytic outside $|z| = 1$ can coincide with $f_1(z)$ within $|z| < 1$. $F(z)$ is the unique analytic continuation of $f_1(z)$ into the entire plane.

Not all functions can be continued indefinitely. The extension may be blocked by a barrier of singularities.

It may also happen that the function $F(z)$ obtained by continuation is multivalued. For example, suppose that after repeating the process described above a number of times, the n th circle of convergence partially overlaps the first one. Then the values of the element $f_n(z)$ in the common region may or may not agree with $f_1(z)$. If they do not agree, then the function $F(z)$ is multivalued, and the "path" along which the continuation was made has encircled one or more branch points.

A power series which converges everywhere defines a single-valued analytic function with no singularities in the entire plane (excluding ∞). Such a function is called an *entire function*. Examples are polynomials, e^z , and $\sin z$. A single-valued function which has no singularities other than poles in the entire plane (excluding ∞) is called a *meromorphic function*. Examples are rational functions, that is, ratios of polynomials.

We conclude by mentioning *Liouville's theorem*; if the function $f(z)$ is regular everywhere in the z -plane, including the point at infinity, then $f(z)$ is a constant.

REFERENCES

A very nice treatment of the theory of functions of a complex variable may be found in the two small volumes by Knopp (K4). This subject is treated in many other books, for example, Copson (C8); Whittaker and Watson (W5); Apostol (A5); Nehari (N2); and Titchmarsh (T4).

PROBLEMS

A-1 Describe the mapping produced by the function

$$W(z) = \frac{1}{\sqrt{(z^2 + 1)(z - 2)}}$$

A-2 Describe the mapping produced by the function

$$W(z) = \frac{1}{\sqrt{z - 1 - i\sqrt{2}}}$$

A-3 Which of the following are analytic functions of the complex variable z ?

- (a) $|z|$
- (b) $\operatorname{Re} z$
- (c) $e^{\sin z}$

Use of (3-28) converts (3-29) to

$$\psi(\mathbf{k}, \mathbf{l}) = \int_0^1 du \int \frac{d\Omega}{\{1 + \hat{\mathbf{r}} \cdot [\mathbf{k}u + \mathbf{l}(1-u)]\}^2} \quad (3-30)$$

The solid-angle integral in (3-30) is just $I_2[\mathbf{k}u + \mathbf{l}(1-u)]$, as given by (3-19). Thus

$$\psi(\mathbf{k}, \mathbf{l}) = 4\pi \int_0^1 \frac{du}{1 - [\mathbf{k}u + \mathbf{l}(1-u)]^2}$$

This is an elementary integral, although rather tedious; the answer has the interesting form

$$\psi(\mathbf{k}, \mathbf{l}) = \frac{4\pi}{\sqrt{A^2 - B^2}} \cosh^{-1} \frac{A}{B} \quad (3-31)$$

where

$$A = 1 - \mathbf{k} \cdot \mathbf{l}$$

$$B = \sqrt{(1 - k^2)(1 - l^2)}$$

The proof of (3-31) is left as an exercise (Problem 3-37).

3-3 CONTOUR INTEGRATION

One of the most powerful means for evaluating definite integrals is provided by the theorem of residues from the theory of functions of a complex variable. We shall illustrate this method of contour integration by a number of examples in this section. Before reading this material, the student who does not know the theory of functions of a complex variable reasonably well should review (or learn) certain parts of this theory. These parts are presented in the Appendix of this book to serve as an aid in the review (or as a guide to the study).

The theorem of residues [Appendix, Eq. (A-15)] tells us that if a function $f(z)$ is regular in the region bounded by a closed path C , except for a finite number of poles and isolated essential singularities in the interior of C , then the integral of $f(z)$ along the contour C is

$$\int_C f(z) dz = 2\pi i \sum \text{residues}$$

where \sum residues means the sum of the residues at all the poles and essential singularities inside C .

The residues at poles and isolated essential singularities may be found as follows.

If $f(z)$ has a simple pole (pole of order one) at $z = z_0$, the residue is

$$a_{-1} = [(z - z_0)f(z)]_{z=z_0} \quad (3-32)$$

If $f(z)$ is written in the form $f(z) = q(z)/p(z)$, where $q(z)$ is regular and $p(z)$ has a simple zero at z_0 , the residue of $f(z)$ at z_0 may be computed from

$$a_{-1} = \frac{q}{p'} \Big|_{z=z_0} \quad (3-33)$$

If z_0 is a pole of order n , the residue is

$$a_{-1} = \frac{1}{(n-1)!} \left\{ \left(\frac{d}{dz} \right)^{n-1} [(z - z_0)^n f(z)] \right\}_{z=z_0} \quad (3-34)$$

If z_0 is an isolated essential singularity, the residue is found from the Laurent expansion (Appendix, Section A-2, item 7).

We illustrate the method of contour integration by some examples.

EXAMPLE

$$I = \int_0^\infty \frac{dx}{1+x^2} \quad (3-35)$$

Consider $\oint dz/(1+z^2)$ along the contour of Figure 3-1. Along the real axis the integral is $2I$. Along the large semicircle in the upper-half plane we get zero, since

$$z = Re^{i\theta} \quad dz = iRe^{i\theta} d\theta \quad \frac{1}{1+z^2} \approx \frac{e^{-2i\theta}}{R^2}$$

$$\int \frac{dz}{1+z^2} \approx \frac{i}{R} \int e^{-i\theta} d\theta \rightarrow 0 \text{ as } R \rightarrow \infty$$

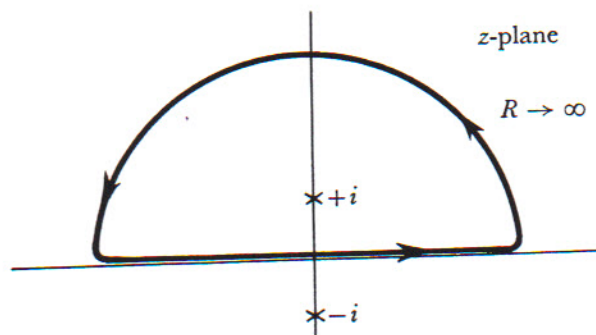


Figure 3-1 Contour for the integral (3-35)

at $z = z_0$, the residue is

$$\left. \frac{1}{(z - z_0)^n} f(z) \right|_{z=z_0} \quad (3-32)$$

, where $q(z)$ is regular and $p(z)$ at z_0 may be computed from

$$(3-33)$$

$$-z_0)^n f(z) \Big|_{z=z_0} \quad (3-34)$$

the residue is found from the item 7).

ation by some examples.

$$(3-35)$$

of Figure 3-1. Along the real
nicircle in the upper-half plane

$$\frac{1}{1+z^2} \approx \frac{e^{-2i\theta}}{R^2}$$

0 as $R \rightarrow \infty$

z -plane

$R \rightarrow \infty$

the integral (3-35)

The residue of $1/(1+z^2) = 1/(z+i)(z-i)$ at $z=i$ is $1/(2i)$. Thus

$$2I = 2\pi i \left(\frac{1}{2i} \right) = \pi \quad I = \frac{\pi}{2}$$

Note that an important part of the problem may be choosing the "return path" so that the contribution from it is simple (preferably zero).

EXAMPLE

Consider a resistance R and inductance L connected in series with a voltage $V(t)$ (Figure 3-2). Suppose $V(t)$ is a voltage impulse, that is, a very high pulse lasting for a very short time. As we shall see in Chapter 4, we can write to a good approximation

$$V(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

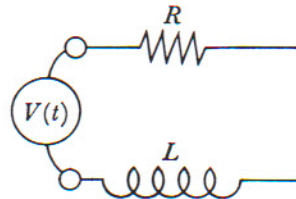


Figure 3-2 Series R-L circuit

where A is the area under the curve $V(t)$.

The current due to a voltage $e^{i\omega t}$ is $e^{i\omega t}/(R + i\omega L)$. Thus the current due to our voltage pulse is

$$I(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{R + i\omega L} \quad (3-36)$$

Let us evaluate this integral.

If $t < 0$, the integrand is exponentially small for $\text{Im } \omega \rightarrow -\infty$, so that we may complete the contour by a large semicircle in the *lower*-half ω -plane, along which the integral vanishes.³ The contour encloses no singularities, so that $I(t) = 0$.

If $t > 0$, we must complete the contour by a large semicircle in the *upper*-half plane. Then

$$I(t) = 2\pi i \left(\frac{A}{2\pi} \right) \frac{e^{-Rt/L}}{iL} = \frac{A}{L} e^{-Rt/L}$$

³ A rigorous justification of this procedure is provided by *Jordan's lemma*; see Copson (C8) p. 137 for example.

EXAMPLE

$$I = \int_0^{\infty} \frac{dx}{1+x^3} \quad (3-37)$$

The integrand is not even, so we cannot extend it to $-\infty$. Consider the integral

$$\int \frac{\ln z \, dz}{1+z^3}$$

The integrand is many-valued; we may cut the plane as shown in Figure 3-3, and define $\ln z$ real ($=\ln x$) just above the cut. Then $\ln z = \ln x + 2\pi i$ below the cut, and integrating along the indicated contour,

$$\oint \frac{\ln z \, dz}{1+z^3} = -2\pi i I$$

On the other hand, using the method of residues,

$$\oint \frac{\ln z \, dz}{1+z^3} = -\frac{4\pi^2 i \sqrt{3}}{9}$$

Thus $I = (2\pi\sqrt{3})/9$.

When integrating around a branch point, as in this example, it is necessary to show that the integral on a vanishingly small circle around the branch point is zero. In this example, this part goes like $r \ln r$, which approaches zero as $r \rightarrow 0$.

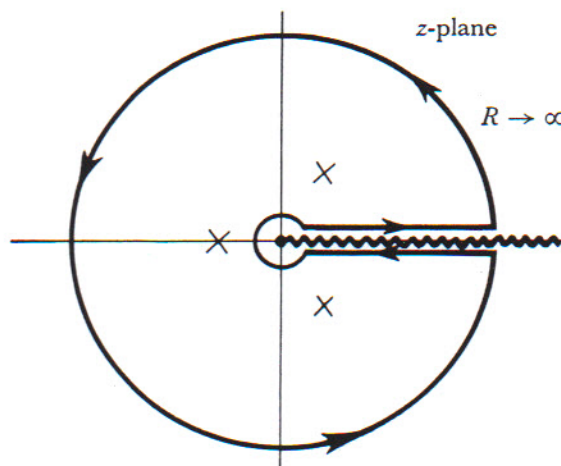


Figure 3-3 Contour for the integral (3-37)